(Active) structural completeness for small frames

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(active) structural completeness

L - a logic, $\vdash_L$ - its consequence relation of derivability
r - a rule
$\vdash^r_L$ - the least consequence relation containing the rule $\{r\} \cup \vdash_L$

r is admissible in L if Theorems$(\vdash^r_L) = L$

$r = \Gamma/\varphi$ is active in L if there is a substitution σ such that $\sigma(\Gamma) \subseteq L$

L is (actively) structurally complete if every (active) admissible in L rule is derivable, i.e., is in $\vdash_L$

Fact

Γ/φ is admissible for $\vdash$ iff $(\forall\gamma \in \Gamma, \vdash \sigma(\gamma))$ yields $\vdash \sigma(\varphi)$ for every substitution σ

(quasi)varieties

identities look like $(\forall \bar{x})\ s(\bar{x}) \approx t(\bar{x})$
quasi-identities look like

$$(\forall \bar{x})\ s_1(\bar{x}) \approx t_1(\bar{x}) \wedge \cdots \wedge s_n(\bar{x}) \approx t_n(\bar{x}) \ \rightarrow\ s(\bar{x}) \approx t(\bar{x})$$

(quasi)varieties = classes of algebras defined by (quasi-)identities

Fact

Every (quasi)variety $\mathcal{V}$ has a free algebra $\mathbf{F}_{\mathcal{V}}(k)$ of rank $k > 0$.
Let $\mathbf{F}_{\mathcal{V}} = \mathbf{F}_{\mathcal{V}}(\aleph_0)$.

Mal'cev

A class is $\mathrm{SPP_U}$-closed iff it is a quasivariety.

Birkhoff

A class is HSP-closed iff it is a variety.

admissibility algebraically

logic L	$\leftrightsquigarrow$	variety $\mathcal{V}$
logical connectives	$\leftrightsquigarrow$	basic operations
theorems	$\leftrightsquigarrow$	identities valid in $\mathcal{V}$
derived rules	$\leftrightsquigarrow$	quasi-identities valid in $\mathcal{V}$
admissible rules	$\leftrightsquigarrow$	quasi-identities valid in $\mathbf{F}_{\mathcal{V}}$
active rules	$\leftrightsquigarrow$	quasi-identities with the premise satisfiable in $\mathbf{F}_{\mathcal{V}}$

Thus we *may* study admissibility and (A)SC for varieties

SC vs ASC

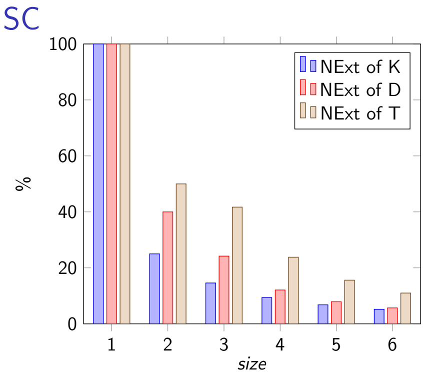
Examples

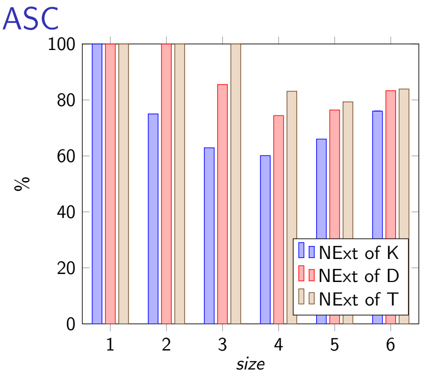
- S5 and Ł$_n$ are ASC but not SC ($n \geqslant 3$) [folklore];
- discriminator varieties are ASC [Burris '92, Dzik '11], and are SC iff they are minimal or trivial (if there are two distinct constants) [Campercholi, S., Vaggione '16];
- ASC normal extensions of S4 are SC iff they extend S4.McKinsey [Dzik and S. '16];
- among 3330 3-element groupoids (up to izo.) 2676 generate SC quasivarieties and 2930 generate ASC quasivarieties [Metcalfe and Röthlisberger '13];
- almost all finite algebras generate SC varieties [Murskiĭ '75].

aim

- To compare ASC and SC for normal modal logics of small frames.
- to understand (A)SC.

numerical results: normal extensions of K

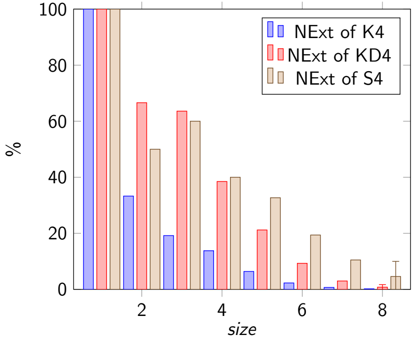




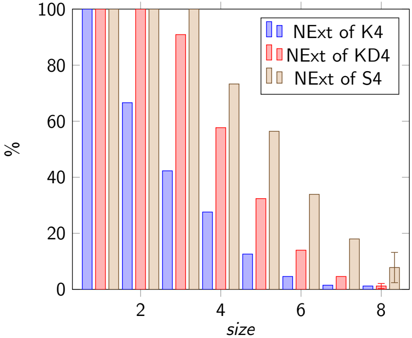
size	1	2	3	4	5	6
K	2	8	86	2838	285799	96420781
D=K$\oplus\Diamond\top$	1	5	62	2214	244134	87722854
T=K$\oplus\Box p \to p$	1	2	12	189	9175	1523497

numerical results: normal extensions of K4

SC



ASC



size	1	2	3	4	5	6	7	8
K4	2	6	26	145	1050	9917	121496	1958413
KD4=K4 $\oplus \Diamond \top$	1	3	11	52	315	2496	26314	370304
S4=K4 $\oplus \Box p \to p$	1	2	5	15	55	242	1322	9160

How it is computed?

decidability

(A)SC-problem for varieties

INPUT: a finite algebra $\mathbf{A}$,
OUTPUT: YES if HSP($\mathbf{A}$) is (A)SC, NO otherwise.

Theorem (Dywan '78, Bergman '88, Metcalfe & Röthlisberger '13, S.'18)

There are algorithms which solve the (A)SC-problem for varieties when the input is from

- a congruence meet-semidistributive variety,
- a congruence modular variety.

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Drawback

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Hope

The algorithm is just very slow in case of congruence distributivity.

SI algebras

An algebra $\mathbf{A}$ is subdirectly irreducible if there is a pair $a, b \in A$ of distinct elements such that every nontrivial congruence of $\mathbf{A}$ contains (a, b).

Fact

An algebra $\mathbf{A}$ is SI if and only if whenever $\mathbf{A} \leqslant \prod \mathbf{A}_i$, then one of the projections $\pi_i : \mathbf{A} \to \mathbf{A}_i$ is an embedding.

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Theorem (Birkhoff '35)

Every variety $\mathcal{V}$ is generated as a quasivariety by its SI algebras:

$$\mathcal{V} = \mathsf{SP}(\mathcal{V}_{SI}).$$

algorithm algebraically

Theorem (Bergman '88, Metcalfe & Röthlisberger '13, Dzik & S. '16)

Let $\mathbf{A}$ be a finite k-generated algebra, $\mathcal{V} = \mathsf{HSP}(\mathbf{A})$ have only finite SI algebras, and $\mathbf{M} \leqslant \mathbf{F}_{\mathcal{V}}$. Then

- $\mathcal{V}$ is SC iff $\mathcal{V}_{SI} \subseteq \mathsf{S}(\mathbf{F}_{\mathcal{V}}(k))$,
- $\mathcal{V}$ is ASC iff $\{\mathbf{S} \times \mathbf{M} : \mathbf{S} \in \mathcal{V}_{SI}\} \subseteq \mathsf{SP}(\mathbf{F}_{\mathcal{V}}(k))$.

Jónsson's Lemma

Let $\mathbf{A}$ be a finite algebra and $\mathcal{V} = \mathsf{HSP}(\mathbf{A})$ be congruence distributive. Then $\mathcal{V}_{SI} \subseteq \mathsf{HS}(\mathbf{A})$ is not too big.

finite duality, a logarithmic reduction

finite modal algebra **A**	$\leftrightsquigarrow$	finite modal frame **W**
HSP(**A**)	$\leftrightsquigarrow$	$L(\mathbf{W})$
subalgebras	$\leftrightsquigarrow$	p-morphic images
homomorphic images	$\leftrightsquigarrow$	generated subframes
products	$\leftrightsquigarrow$	disjoint union
finite SI algebras	$\leftrightsquigarrow$	finite rooted frames
free algebra $\mathbf{F}(k)$	$\leftrightsquigarrow$	universal model $\mathbf{U}(k)$

$$|W| = \log_2 |A|, \quad |U(k)| = \log_2 |F(k)|$$

algorithm relationally I

Corollary

Let $\mathbf{W}$ be a modal frame and $L = L(\mathbf{W})$ be its logic. Let k be a smallest number for which there exists a k-valution v such that the model $(\mathbf{W}, v)$ has only trivial bisimulations. Then

- L is SC iff every rooted gen. subframe of $\mathbf{W}$ is a p-morphic image of $\mathbf{U}(k)$,
- L is ASC iff $\mathbf{R} \sqcup \mathbf{U}(0)$ is a p-morphic image of a disjoint union of copies of $\mathbf{U}(k)$ for every rooted gen. subframe $\mathbf{R}$ of $\mathbf{W}$.

Remark

It is still slow: $k \leqslant \log_2 |W|$ and just $|U(k)| \leqslant |W| \cdot 2^{|W| \cdot \log_2 |W|}$. so the algorithms works in 2EXPTIME.

we do not need to check SC

Fact (S. & U. '18)

Let **W** be a finite frame and $L = L(\mathbf{W})$ be its logic. Assume that L is ASC. Then L is SC iff

- L is serial ($\Diamond 1 \in L$) and every top cluster in **W** consists of one reflexive point.

  or
- $L = Ver$ (**W** has empty accessibility relation).

proof

"The same" as for normal extensions of S4. One just need to consider *weak* transitivity and *weak* McKinsey's law.

we do not need to check non-serial frames

Fact (S. & U. '18)

Let $\mathbf{W}$ be a finite frame. Then $\mathbf{W}$ is ASC iff $\mathbf{W} \cong \mathbf{W}_{ser} \sqcup \mathbf{W}_{ver}$, where $\mathbf{W}_{ser} \models \Diamond 1$, $L(\mathbf{W}_{ser})$ is ASC and $\mathbf{W}_{ver} \models \Box 0$,

Remark: It works without the finiteness assumption but the (algebraic) proof we have for it is harder.

proof

By considering the structure of $\mathbf{U}(k)$ and p-morphisms onto $\circ \sqcup \mathbf{U}'(0)$, where $\mathbf{U}'(0)$ is a gen. subframe of $\mathbf{U}(0)$.

structure of $\mathbf{U}(k)$

Let $\mathbf{W}$ be an input frame.
Let $Val(k) = (2^W)^{\{p_1,\dots,p_k\}}$ be the set of all k-valuations of $\mathbf{W}$.
Then $\mathbf{U}(k)$ is the underlying frame of the model

$$\left(\bigsqcup_{w \in Val(k)} (\mathbf{W}, w) \right) /\beta,$$

where β is a largest bisimulation

$$(x, y) \in \beta \quad \text{iff} \quad \text{the same } k\text{-formulas are satisfied in } x \text{ and } y.$$

algorithm relationally II

Corollary

Let $\mathbf{W}$ be a serial frame and $L = L(\mathbf{W})$ be its logic. Let k be a smallest number for which there exists a k-valution v such that the model $(\mathbf{W}, v)$ has only trivial bisimulations. Then

- L is ASC iff $\mathbf{R}$ or $\mathbf{R} \sqcup \bullet$ is a p-morphic image of $\mathbf{U}(k)$ for every rooted gen. subframe $\mathbf{R}$ of $\mathbf{W}$.

Remark: Still quite slow, though enough for 5-element frames.

basic idea for improvement

Observation (Metcalfe & Röthlisberger '13)

Let $\mathbf{U}^p(k)$ be frame such that

- $\mathbf{W}$ embeds as a gen. subframe into $\mathbf{U}^p(k)$,
- $\mathbf{U}^p(k)$ is a p-morphic image of $\mathbf{U}(k)$.

Then in the algorithm we may replace $\mathbf{U}(k)$ for $\mathbf{U}^p(k)$.

proof

The duals of $\mathbf{U}(k)$ and $\mathbf{U}^p(k)$ generate the same quasivariety.

How to find a small $\mathbf{U}^p(k)$?

structure of $\mathbf{U}^p(k)$

Recall that $\mathbf{U}(k)$ is the underlying frame of the model

$$\left(\bigsqcup_{v \in Val(k)} (\mathbf{W}, w) \right) / \beta,$$

where β is a largest bisimulation
The frame $\mathbf{U}^p(k)$ is of the form

$$\left(\bigsqcup_{w \in Val(k)} \mathbf{W} \right) / \gamma,$$

where γ is a frame bisimilar equivalence extending β and not gluing elements from a chosen copy of W.

optimalization ingredients

1. Do not compute $\mathbf{U}(k)$ at all!
2. Search p-morphisms reasonably?
3. Use randomness (Las Vegas method)!

sample reduction

let $(\mathbf{W}_i, w_i)$, $i \leqslant N$ be the list of all k-models based on (copies of) $\mathbf{W}$

Put $(\mathbf{V_0}, v_0) = (\mathbf{W_0}, w_0)$,
Once defined $(\mathbf{V}_i, v_i)$: Let, say

$$w'_{i+1}(x) = \begin{cases} w_{i+1}(x) & \text{if } x \text{ is bisimilar to } y \text{ in } (\mathbf{W}_0, w_0) \\ \emptyset & \text{in the oposite case} \end{cases}$$

and take

$$\mathbf{V}_{i+1} = (\mathbf{V}_i, v_i) \sqcup (\mathbf{W}_{i+1}, w'_{i+1})/(\text{a largest bisimulation})$$

and dafine $\mathbf{U}^p(k) = \mathbf{V}_N$.

Remarks:

- More optimalizations are used, but this one is the most efficient.
- We incorporate ramdomness here.
- It is sufficient for 6-elements frames

do we really need this algorithm?

- Find an easy to check condition suffitient for $\neg$ASC!
- Find an easy to check condition suffitient for ASC!

condition for $\neg$ASC

Let $\mathbf{R} \sqsubseteq \mathbf{S}$ iff there is a surjective p-morphism $\mathbf{S} \to \mathbf{R}$. Let $\mathcal{M}(\mathbf{W})$ be the set of generated rooted subframes of $\mathbf{W}$ which are maximal w.r.t. $\sqsubseteq$.

Fact (S. & U. '19)

If some $\mathbf{R} \in \mathcal{M}(\mathbf{W})$ is a proper gen. subframe of a rooted gen. subframe of $\mathbf{W}$, then $L(\mathbf{W})$ is not ASC.

proof

Similar as we deal with non-serial frames.

Remarks:

- It is easy to be check.
- It covers around 99% of $\neg$ASC frames we checked.

condition for ASC

Observation (Dzik '11)

If L admits a projective unification, then L is ASC.

Corollary

If the transitive closure of the accessibility relation of $\mathbf{W}$ is symmetric, then $L(\mathbf{W})$ is ASC.

proof

The corresponding variery is discriminator. By Burris' result, it admits projective unification.

Theorem (Dzik & Wojtylak '12, Kost '18)

There is a simple characterization of transitive frames which logics admits projective unification.

The end

This is all Thank you!